Department of Applied Mathematics

Preliminary Examination in Numerical Analysis

August 17, 2016 , 10 am – 1 pm.

Submit solutions to four (and no more) of the following six problems. Show all your work, and justify all your answers. No calculators allowed.

Problems and solutions given below:

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1. Root finding / Nonlinear equations

Consider the scalar equation $F(x) \quad 0$. Assume   is a root of the equation.

a. Give the recursion for the Newton method for approximating a root.

b. Give conditions on $F(x)$ near

Solution: Nonlinear Equations

(a) Newton:

$$x_{j+1} = x_j - \frac{F(x_j)}{F'(x_j)} \qquad (1)$$

(c) Note that $F(\alpha) = 0$ and solve for α to get

$$\alpha = x - \frac{F(x)}{F'(x)} - \frac{1}{2}\frac{F''(\eta)}{F'(x)}(\alpha - x)^2.$$

$$\alpha - x_{j+1} = -\frac{1}{2}\frac{F''(\eta)}{F'(x)}(\alpha - x_j)^2.$$

$$M|\alpha - x_1| \le (M|\alpha - x_0|)^2,$$

which implies $M|\alpha - x_1| < M|\alpha - x_0|$, and, thus, $x_1 \in \mathcal{N}$. By induction, this also implies $x_j \in \mathcal{N}$ for $j > 1$ and

Thus $x_j \to \alpha$ and convergence is quadratic.

2.

3. Interpolation / Approximation

a. Define what is meant by *cubic splines* and, for these, *natural* and *not-a-knot* conditions.

b. Determine the *not-a-knot* cubic spline $s(x)$ that satisfies the data

x	1	0	1	2
y	2	3	4	1

.

c. If, at the nodes $x \quad h, 0, h$, one has function values $y_{\ h}, y_0, y_h$ and forms a quadratic interpolant $s(x)$, one obtains $s'(0) \quad [\ \ \frac{1}{2} y_{\ h} \quad \frac{1}{2} y_h] / h$, i.e. the finite difference weights can be written as $[\ \ \frac{1}{2}, 0, \ \ \frac{1}{2}] / h.$ It might be tempting to replace the quadratic interpolant here with a natural cubic spline (hoping to increase the approximation's order of accuracy). Work out the weights you get in this case.

Solution:

a. A *cubic spline* is a cubic polynomial between adjacent nodes, and features continuous function, first and second derivative at the nodes – i.e. the third derivative may be discontinuous at the nodes. Without additional end conditions, a cubic spline will have two free parameters. A *natural* cubic spline adds the two extra conditions that $s''(x) \quad 0$ at each end point. The *not-a-knot* cubic spline instead removes two 'freedoms', i.e. the cubic spline is not allowed to have a jump in its third derivative one node point in from each boundary.

b. With four node points, and jumps in the third derivative not allowed at either of the two internal nodes, the spline becomes a single cubic, i.e. we can immediately find it, for ex., by Lagrange's or Newton's interpolation formulas. Choosing, for ex., the Newton approach, the divided difference table becomes

-1	-2			
		-1		
0	-3		0	
		-1		1
1	-4		3	
		5		
2	1			

from which we read off the polynomial as $s(x) \quad 2 \quad 1\ (x \quad 1) \quad 0\ (x \quad 1)x \quad 1\ (x \quad 1)x(x \quad 1)$ $x^3 \quad 2x \quad 3.$

c. Since the spline $s(x)$ is not discontinuous at $x \quad 0$ until in the third derivative, we can write it:

$$[\ \ h, 0] \qquad a \quad bx \quad cx^2 \quad dx^3$$

$$[0, \ \ h] \qquad a \quad bx \quad cx^2 \quad ex^3$$

The natural end conditions give $2c \quad 6dh \quad 0$ and $2c \quad 6eh \quad 0$, resp., i.e. $e \qquad d.$ Enforcing the values at the nodes now give

$$a \quad bh \quad ch^2 \quad dh^3 \qquad y_{\ h}$$

$$a \qquad y_0$$

$$a \quad bh \quad ch^2 \quad dh^3 \qquad y_h$$

Subtracting the top equation from the bottom one gives $2bh \quad y_h \quad y_h$, and we obtain the same approximation for $s'(0)$

4. **Linear Algebra**

Consider the linear system $A\underline{x} \quad \underline{b}$, where $A_{n\ m}, \underline{x}_{m\ 1}, \underline{b}_{n\ 1}$.

a. Describe the three possible cases for existence and uniqueness of a solution of the linear system. Give criteria on $A, \underline{b}$ that distinguish each case.

b. Let $\underline{x}_{LS}$ be a minimizer of the least squares functional, that is, let

$$\| A\underline{x}_{LS} \quad \underline{b} \|_2 \quad \min_{\underline{x}} \| A\underline{x} \quad \underline{b} \|_2 .$$

(i) Does $\underline{x}_{LS}$ always exist? Explain your answer.

(ii) Give conditions on $A, \underline{b}$ such that $\underline{x}_{LS}$ is unique.

(iii) In the case of a unique solution, give an expression for the least squares solution $\underline{x}_{LS}$.

(iv) ITc[(4t43 -2.isol3(x)T32 Tc4r nu)-.695.e lean o]TJ presstoast squares solutioproe

Solutions: Linear Algebra

(a) The three cases are:

There exists a unique solution if: (i) The columns of A are linearly independent (every column is a pivot column), (ii) $b \in$ Range of A.

There exists an infinite number of solutions if: (i) $b \in$ Range of A, (ii) the columns of A are linearly dependent.

There exists zero solutions if: $b \notin$ Range of A.

(b)

(c) The pseudo-inverse can be computed by using the singular value decomposition (SVD)

$$A = U\Sigma V^*,$$

where $U_{n\times n}, V_{m\times m}$ are unitary and $\Sigma_{n\times m}$ is diagonal:

$$\Sigma_{n\times m} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

and σ_j are the singular values. Then,

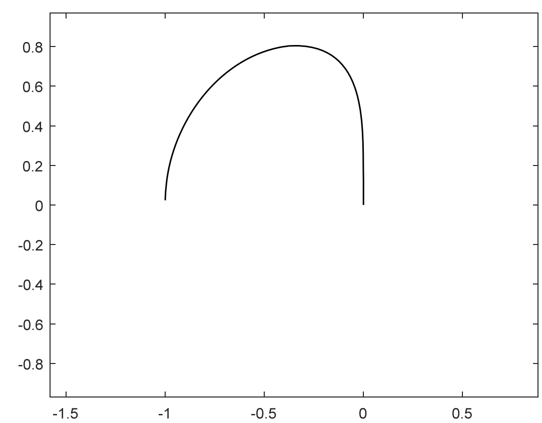
$$(A^*A)^\dagger A^* = A^\dagger = V\Sigma^\dagger U^*,$$

where

$$\Sigma^\dagger_{m\times n} = \mathrm{diag}\{..., \sigma_j^\dagger, ...\}$$

and

$$\sigma_j^\dagger = \begin{cases} \frac{1}{\sigma_j} & \sigma_j \neq 0 \\ 0 & \sigma_j = 0 \end{cases}$$


0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.5
-1
-0.5
0
0.5

Solution: PDF

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